Linearity of Determinants

given row vectors $\vec{v}_1, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_n$ each has n components

then $T(\vec{x}) = \det \begin{bmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_{i-1} \\ \vec{x} \\ \vec{v}_{i+1} \\ \vdots \\ \vec{v}_n \end{bmatrix} : \mathbb{R}^{1\times n} \longrightarrow \mathbb{R}$ is a LT

next, use properties: $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$
$T(k\vec{x}) = kT(\vec{x})$

no parens

likewise, $\det \begin{bmatrix} \vec{v}_1 \\ \vdots \\ \vec{x}+\vec{y} \\ \vdots \\ \vec{v}_n \end{bmatrix} = \det \begin{bmatrix} \vec{v}_1 \\ \vdots \\ \vec{x} \\ \vdots \\ \vec{v}_n \end{bmatrix} + \begin{bmatrix} \vec{v}_1 \\ \vdots \\ \vec{y} \\ \vdots \\ \vec{v}_n \end{bmatrix}$

and $\det \begin{bmatrix} \vec{v}_1 \\ \vdots \\ k\vec{x} \\ \vdots \\ \vec{v}_n \end{bmatrix} = k \det \begin{bmatrix} \vec{v}_1 \\ \vdots \\ \vec{x} \\ \vdots \\ \vec{v}_n \end{bmatrix}$

given $A = \begin{array}{c} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \end{array}\begin{array}{c} \text{I} \\ \text{II} \end{array}$ know $\det A = ad - bc$

$\therefore$ given $B = \begin{bmatrix} a/k & b/k \\ c & d \end{bmatrix}$ then $\det B = \frac{a}{k}d - \frac{b}{k}c$

$= \frac{1}{k}(ad - bc) = \frac{1}{k}\det A$

also, $B = A \to \text{II} \cdot \frac{1}{k}$

$= \begin{bmatrix} a & b \\ \frac{c}{k} & \frac{d}{k} \end{bmatrix}$ then $\det B = \frac{1}{k}\det A$

swap rows:

1 inversion of A

$\begin{bmatrix} k & k \end{bmatrix}$

swap rows:

if $B = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$ then $\det B = (-1)^{1} \det A$ ← 1 inversion of A

$k(c+d) = kc + kd$ add I of A:

then $B = \begin{bmatrix} a+kc & b+kd \\ c & d \end{bmatrix}$ then $\det B = (a+kc)d - c(b+kd)$

$= ad + \cancel{kcd} - bc - \cancel{kcd}$

$= \det A$

generalize:

- if B is obtained by dividing a row of A by a scalar k then $\det B = \frac{1}{k} \det A$
- if B is obtained from A by a row swap then $\det B = -\det A$ ← "determinant is alternating on rows"
- if B is obtained from A by adding a multiple of A to another row then $\det B = \det A$

Furthermore, if A, B are $n \times n$ matrices and $m \in \mathbb{Z}^{+}$ then:

- $\det(AB) = \det A \det B$
- $\det(A^{m}) = (\det A)^{m}$

also $A \sim B \rightarrow \det A = \det B$

also if A is invertible (or $\det A \neq 0$) then $\det(A^{-1}) = \frac{1}{\det A} = (\det A)^{-1}$

reciprocal

proof:

given A is invertible $\Rightarrow I_n = AA^{-1}$

$\det(I_n) = \det(AA^{-1})$

$\frac{1}{\det A} = \frac{\det A}{\det A} \det A^{-1}$

$\frac{1}{\det A} = \det A^{-1}$ $\square$

§6.3 Determinant Applications

recall: given linear system $\left| \begin{matrix} 2x_1 - x_2 = 3 \\ x_1 + 2x_2 = 1 \end{matrix} \right|$ can be written as $\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$$A \vec{x} = \vec{b}$$

$$\vec{x} = A^{-1} \vec{b}$$

$\det A = 4 - (-1) = 5$

$$= \frac{1}{\det A} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 7 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7/5 \\ -1/5 \end{bmatrix}$$

Generalize:

given $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

then $A^{-1} = \dfrac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

$$A^{-1} \vec{b} = \frac{1}{\det A} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

row 1 $\underset{a\,d\;-\;b\,c}{a_{22} b_1 - a_{12} b_2} = \det \underbrace{\begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix}}_{A_{\vec{b},1}}$; row 2 $\underset{-b\,c\;+\;a\,d}{-a_{21} b_1 + a_{11} b_2} = \det \underbrace{\begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix}}_{A_{\vec{b},2}}$

<u>Cramer's Rule</u>: solution of system $A\vec{x} = \vec{b}$ is:
(assume A is 2×2 invertible matrix)

$$x_1 = \frac{\det [A_{\vec{b},1}]}{\det A}, \quad x_2 = \frac{\det [A_{\vec{b},2}]}{\det A}$$

revisit: solve $\underset{A}{\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}$ $\det A = 5$

$A_{\vec{b},1} = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}$ $\det A_{\vec{b},1} = 6 + 1 = 7$

$$x_1 = \frac{\det A_{\vec{b},1}}{\det A} = \frac{7}{5}$$

$$\vec{x} = \begin{bmatrix} 7/5 \\ -1/5 \end{bmatrix}$$

$A_{\vec{b},2} = \begin{bmatrix} 2 & 3 \ldots$ $\det A \ldots 2 - 3 = -1$

$$A_{\vec{b},2} = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \qquad \det A_{\vec{b},2} = 2-3 = -1$$

$$x_2 = \frac{\det A_{\vec{b},2}}{\det A} \rightarrow \frac{-1}{5}$$

$$\vec{x} = \begin{bmatrix} \\ -1/5 \end{bmatrix}$$

ex solve the system
$$\left| \begin{array}{l} x - y + 2z = -3 \\ 3x \quad\;\; + z = -2 \\ -x + 2y \quad\;\; = -2 \end{array} \right|$$

write as $A\vec{x} = \vec{b}$:
$$\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ -2 \end{bmatrix}$$

$$\det A = 1 \det\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} - (-1) \det\begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix} + 2 \det\begin{bmatrix} 3 & 0 \\ -1 & 2 \end{bmatrix}$$

$$= -2 + 1 + 2(6) = 11 = \det A$$

$$\det(A_{\vec{b},1}) = \det\begin{bmatrix} -3 & -1 & 2 \\ -2 & 0 & 1 \\ -2 & 2 & 0 \end{bmatrix} = 0 \qquad x = \frac{0}{11} = 0$$

$$\det(A_{\vec{b},2}) = \det\begin{bmatrix} 1 & -3 & 2 \\ 3 & -2 & 1 \\ -1 & -2 & 0 \end{bmatrix} = -11 \qquad y = \frac{-11}{11} = -1$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}$$

$$\det(A_{\vec{b},3}) = \det\begin{bmatrix} 1 & -1 & -3 \\ 3 & 0 & -2 \\ -1 & 2 & -2 \end{bmatrix} = -22 \qquad z = \frac{-22}{11} = -2$$

<u>Cramer's Rule</u> – with $n \times n$ invertible matrix A

given $A\vec{x} = \vec{b}$

Components of solution vector $\vec{x}$ are: $x_i = \dfrac{\det(A_{\vec{b},i})}{\det A}$

where $A_{\vec{b},i}$ is obtained by replacing the ith column of A by $\vec{b}$

but what <u>IS</u> a determinant???

recall: $n \times n$ matrix A is orthogonal iff $A^T A = I_n$
or
$A^{-1} = A^T$

also recall: given $n \times n$ matrices A, B

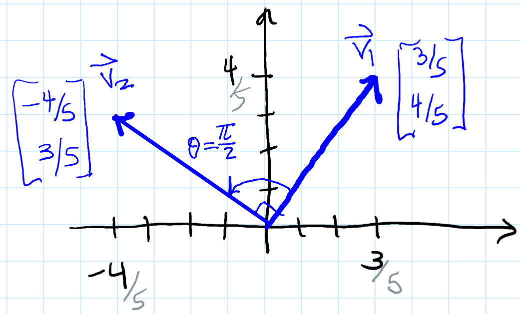
$$\det A^T = \det A$$
$$\det(AB) = \det A \det B$$
$$\det A^m = (\det A)^m \qquad m \in \mathbb{Z}^+$$

$$A^T A = I_n$$
$$\det(A^T A) = \det I_n$$
$$\det A^T \det A = 1$$
$$\det A \det A$$
$$(\det A)^2 = 1 \Rightarrow \det A = \pm 1$$

thm: determinant of orthogonal matrix is either 1 or -1

ex. use CCW $\pi/2$ rotation matrix: $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

$$\det \begin{array}{c} \begin{bmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{bmatrix} \\ \begin{array}{cc} \vec{v}_1 & \vec{v}_2 \end{array} \end{array} = \frac{9}{25} - \left(-\frac{16}{25}\right) = 1$$



def'n: orthogonal $n \times n$ matrix A with $\det A = 1$ is a rotation matrix